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THE CARTESIAN SQUARE OF THE HOROCYCLE FLOW IS NOT LOOSELY BERNOULLI

BY

MARINA RATNER†

ABSTRACT

We give an example of an algebraic non-loosely Bernoulli flow. Namely, we prove that the cartesian square of the classical horocycle flow is not loosely Bernoulli.

Loose Bernoulliness (LB) has been introduced into ergodic theory by J. Feldman [4] and A. B. Katok [6] (see also [15]). In his paper [4] J. Feldman gave an example of a non-LB transformation. D. Rudolph [12] then constructed uncountably many non-Kakutani equivalent transformations of both zero and positive entropy. He and D. Ornstein [13] also constructed an LB mixing T s.t. $T\times T$ is not LB (see [14] for loose Bernoulliness of cartesian products). A. B. Katok [6] was the first to construct a non-LB ergodic C^∞-diffeomorphism on a compact C^∞-manifold preserving a smooth measure.

From a certain point of view all these examples are "unnatural" since they all are based on an artificial method of construction (see (B), page 37, in J. Feldman's paper [4]).

In this paper we give an example of a "natural" non-LB ergodic flow, i.e. a non-LB algebraic (and therefore analytic) flow on a homogeneous (and therefore analytic) compact space preserving Haar measure (which is a smooth measure).

More precisely, let

$$G=\mathrm{SL}(2,R),\qquad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}=e$$

and $\Gamma\subset G$ be a discrete subgroup of G s.t. Γ/G is compact. We assume that $-I\in\Gamma$ and if $A\in\Gamma$, $A\neq I$, $-I$ then A is hyperbolic, i.e. $|\mathrm{Tr}\,A|>2$. $M=\Gamma/G$ can be viewed as the space of unit tangent vectors (linear elements) to a compact Riemann surface of constant negative curvature (see [1], [2]).

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The classical horocycle flow h_t is a flow on M defined by

$$h_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad g \in G.$$

h_t preserves the Riemannian volume μ on M (derived from the Haar measure $\tilde{\mu}$ on G), is mixing of all degrees [8], and has zero-entropy. It was shown in [11] that h_t (and therefore h_1 [15]) is LB.

We denote $Y = M \times M = \Gamma \times \Gamma / G \times G$ and $S_t = h_t \times h_t$. Y is a compact homogeneous space and S_t is an algebraic flow on Y preserving the measure $\nu = \mu \times \mu$ (derived from the Haar measure $\tilde{\nu} = \tilde{\mu} \times \tilde{\mu}$ in $G \times G$). S_t is ergodic and of zero entropy.

We prove the following theorem.

THEOREM 1. *The cartesian square $S_t = h_t \times h_t$ of the classical horocycle flow h_t is not loosely Bernoulli.*

We shall also consider the classical geodesic flow g_t on M defined by

$$g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad g \in G$$

(see [1], [2], [5]). g_t also preserves μ and

$$g_t \circ h_u = h_{e^{2t}u} \circ g_t, \qquad u, t \in R.$$

Let $\bar{g}_t = g_t \times g_t$. Then $\bar{g}_t$ preserves ν on Y and

$$\bar{g}_t \circ S_u = S_{e^{2t}u} \circ \bar{g}_t, \qquad u, t \in R. \tag{1}$$

(1) implies via [8] that in fact S_t is mixing of all degrees.

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1. The $\bar{f}$-metric (see [10], [15])

Let $w, w' \in \{1, 2, \cdots, a\}^n$. Then $\bar{f}_n(w, w') = 1 - k/n$ where k is the maximal integer for which we can find subsequences $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_k$ with $w(i_r) = w'(j_r)$, $1 \leqq r \leqq k$.

Let T be an m.p.t. in a probability space (X, m) and let $P = \{P_1, \cdots, P_a\}$ be a partition of X. If $x \in P_j$ then j is the P-name of $x \in X$. Denote $w_n(x) = \{x_0, \cdots, x_n\}$ where x_i is the T^{-i} P-name of x.

Let T have zero-entropy. P is called an LB-partition for T if given $\varepsilon > 0$ there is $N > 0$ and a set $Y \subset X$, $m(Y) > 1 - \varepsilon$ s.t. if $n \geqq N$ and $x, y \in Y$ then $\bar{f}_n(w_n(x), w_n(y)) < \varepsilon$. We say that $w_n(x)$ and $w_n(y)$ or $\{x, Tx, \cdots, T^n x\}$ and $\{y, Ty, \cdots, T^n y\}$ are ε-P-matchable.

An ergodic T is LB if every finite measurable partition of X is LB.

Let S_t be an m.p. flow in a probability space (Y, ν). By Ambrose–Kakutani Theorem S_t can be represented as a special flow over an m.p.t. ϕ in (X, m) built with a positive integrable function F on X. (X, m, ϕ) is called a cross-section for S_t. S_t is ergodic iff ϕ is and by Abramov's formula S_t has zero entropy iff ϕ does. S_t is called LB if it has an LB cross-section. (Then all cross-sections of S_t are LB [15].)

Let (Y, ν, S_t) be a special flow over (X, m, ϕ) built with $\tilde{F} > F > 0$, $\bar{F} = \int_X F dm < \infty$, $Y = \{(x, t)\colon x \in X, \quad 0 \leqq t < F(x), \quad (x, F(x)) = (\phi x, 0)\}$, $d\nu = dm \times dt / \bar{F}$.

Let $\beta = \{A_1, \cdots, A_a\}$ be a partition of X. Denote

$$P_i = \bigcup_{x \in A_i} \bigcup_{k=0}^{F(x)} S_k(x, 0).$$

$\alpha = \{P_1, \cdots, P_a\}$ is a partition of Y. Let ξ be the partition of Y into orbit intervals $[y, S_{F(x)} y)$ where $y = (x, 0)$, $x \in X$.

For $u \in Y$ we denote $I_t(u)$ the orbit interval $[u, S_t u)$, $t > 0$, $I_t(u)$ can be uniquely represented as a disjoint union of intervals $\bigcup_{i=0}^{i(u)} J_i(u)$, $J_i < J_{i+1}$ where $J_0(u), J_{i(u)}(u) \subset C \in \xi$ and $J_i(u) \in \xi$, $i = 1, 2, \cdots, i(u) - 1$. If $J_i(u) \subset P_k$ for some $P_k \in \alpha$ then we say that k is the α-name of $J_i(u)$.

Definition 1. For $u, v \in Y$ $I_t(u)$ and $I_s(v)$ are called ε-α-matchable if there are subsequences $0 \leqq i_1 < i_2 < \cdots < i_k \leqq i(u)$, $0 \leqq j_1 < j_2 < \cdots < j_k \leqq i(v)$ s.t. $J_{i_p}(u)$ and $J_{j_p}(v)$ have equal α-names, $p = \overline{1, k}$, and the measure $l(\bigcup_{p=1}^k J_{i_p}(u))/t$, $l(\bigcup_{p=1}^k J_{j_p}(v))/s$ are at least $1 - \varepsilon$, where l denotes the Lebesque measure on the orbits of S_t.

The sequences $\{i_p, j_p\}_{p=1}^k$ define an α-match σ between $I_t(u)$ and $I_s(v)$, $\sigma(i_p) = j_p$, $p = \overline{1, k}$. We denote

$$\bar{f}(\sigma) = \frac{1}{2}\left[\frac{l\Big(\bigcup_{i \notin \{i_p\}} J_i(u)\Big)}{t} + \frac{l\Big(\bigcup_{j \notin \{j_p\}} J_j(v)\Big)}{s}\right].$$

σ is an ε-α-match iff $\bar{f}(\sigma) < \varepsilon$. We define $\bar{f}(I_t(u), I_s(v)) = \inf_\sigma \bar{f}(\sigma)$. A pair $B = \{I_a(u), I_b(v)\}$ forms a block if $\bar{f}(I_a(u), I_b(v)) = 0$. If $i_{p+n} = i_p + n$, $j_{p+n} =$

$j_p + n$ for some $1 \leqq p \leqq k$ and all $0 \leqq n \leqq n(p)$ then $\bigcup_{n=0}^{n(p)} J_{i_{p+n}}(u)$ and $\bigcup_{n=0}^{n(p)} J_{i_{p+n}}(v)$ form a block.

LEMMA 1. *Suppose that S_t is* LB *and of zero entropy. Then given $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon) > 0$ and $Y_1 \subset Y$, $\nu(Y_1) > 1 - \varepsilon$ s.t. if $u, v \in Y_1$ and $t > t_0$ then $I_t(u)$ and $I_t(v)$ are ε-α-matchable.*

PROOF. S_t is LB iff ϕ is LB. So the partition β is LB for ϕ. Let $\varepsilon > 0$ be given and let $\delta > 0$ be chosen later. Let $N_1 = N_1(\delta) > 0$ and $X_1 \subset X$, $m(X_1) > 1 - \delta$ be s.t. if $x, y \in X_1$ and $n \geqq N_1$ then $\{\phi x, \phi^2 x, \cdots, \phi^n x\}$ and $\{\phi y, \phi^2 y, \cdots, \phi^n y\}$ are δ-β-matchable.

Since ϕ is ergodic there is $N_2 = N_2(\delta) > 0$ and $X_2 \subset X$, $m(X_2) > 1 - \delta$ s.t. if $x \in X_2$ and $n > N_2$ then

$$\left| \frac{1}{n} \sum_{i=1}^{n} F(\phi^i x) - \bar{F} \right| < \delta$$

or

$$\left| \sum_{i=1}^{n} F(\phi^i x) - n\bar{F} \right| < n\delta. \tag{2}$$

Denote $X_3 = X_1 \cap X_2$, $m(X_3) > 1 - 2\delta$ and

$$Y_1 = \bigcup_{x \in X_3} \bigcup_{k=0}^{F(x)} S_k(x, 0), \nu(Y_1) > 1 - 2\tilde{F}\delta.$$

Let $N_3 = \max\{N_1, N_2, N_1\dot{\tilde{F}}/\bar{F}\}$ and let $t_0 = t_0(\delta) = N_3\bar{F}$.

Let $u, v \in Y_1$ and $t > t_0$. Let $I_t(u) = \bigcup_{i=0}^{i(u)} J_i(u)$, $I_t(v) = \bigcup_{j=0}^{i(v)} J_j(v)$ be the above decompositions into α-named orbit intervals.

We have $u = (x, p)$, $v = (y, q)$ for some $x, y \in X_3$, $0 \leqq p < F(x)$, $0 \leqq q < F(y)$ and if $1 \leqq i < i(u)$, $1 \leqq j < i(v)$ then $J_i(u) = [u_i, u_{i+1}]$, $J_j(v) = [v_j, v_{j+1}]$ where $u_i = (\phi^i x, 0)$, $v_j = (\phi^j y, 0)$. So $l(J_i(u)) = F(\phi^i x)$, $l(J_j(v)) = F(\phi^j y)$.

We have

$$\begin{aligned} &\left| t - l\left(\bigcup_{i=1}^{i(u)-1} J_i(u) \right) \right| < 2\tilde{F}, \\ &\left| t - l\left(\bigcup_{j=1}^{i(v)-1} J_j(v) \right) \right| < 2\tilde{F}. \end{aligned} \tag{3}$$

We have $n = [t/\bar{F}] > N_3 > N_2$, $x, y \in X_3$ and by (2):

$$\left| \sum_{i=1}^{n} F(\phi^i x) - t \right| < 2n\delta = \frac{2t}{\bar{F}}\delta,$$

(4)
$$\left|\sum_{i=1}^{n} F(\phi^i y) - t\right| < \frac{2t}{\bar{F}}\,\delta,$$

for sufficiently large t.

Let $r = \min\{n, i(u)-1, i(v)-1\}$. We get from (3) and (4) that

(5)
$$\left| l\Big(\bigcup_{i=1}^{r} J_i(u)\Big) - t\right| < 10t\delta/\bar{F},$$
$$\left| l\Big(\bigcup_{i=1}^{r} J_i(v)\Big) - t\right| < 10t\delta/\bar{F}.$$

Here $J_i(u) = [u_i, u_{i+1})$, $J_j(v) = [v_j, v_{j+1})$ where $u_i = (\phi^i x, 0)$, $v_j = (\phi^j y, 0)$, $x, y \in X_3$.

We have

$$i(u), i(v) > \frac{t}{\tilde{F}} > \frac{N_3 \bar{F}}{\tilde{F}} > \frac{N_1 \tilde{F}\bar{F}}{\bar{F}\tilde{F}} = N_1$$

and $n > N_3 > N_1$. So $r > N_1$ and therefore $\{\phi x, \cdots, \phi^r x\}$ and $\{\phi y, \cdots, \phi^r y\}$ are δ-β-matchable.

Let $1 \leqq i_1 < \cdots < i_k \leqq r$, $1 \leqq j_1 < j_2 < \cdots < j_k \leqq r$, $1 - k/r < \delta$ be s.t. β-names of $\phi^{i_p} x$ and of $\phi^{j_p} y$, $p = \overline{1,k}$ are equal. Then $J_{i_p}(u)$ and $J_{j_p}(v)$ have equal α-names and

$$l\Big(\bigcup_{i \notin \{i_p\}} J_i(u)\Big),\ l\Big(\bigcup_{j \notin \{j_p\}} J_i(v)\Big) < r\delta\tilde{F} < n\delta\tilde{F} \leqq \frac{t\tilde{F}}{\bar{F}}\,\delta.$$

This and (5) show that $I_t(u)$ and $I_t(v)$ are $K\delta$-α-matchable where $K = \max\{(\tilde{F}+10)/\bar{F}, 2\tilde{F}\}$. We complete the proof picking $\delta \leqq \varepsilon/K$. □

I am grateful to Bob Solovay for his proof of the following lemma.

Let I be an interval and J_i, J_j be disjoint subintervals of I, $J_i = [x_i, y_i]$, $y_i \leqq x_j$. We denote $d(J_i, J_j) = l[y_i, x_j]$.

Lemma 2. *Let C, γ, $r > 0$, $0 < a < 1$ be constants. There is $\theta = \theta(C, \gamma, r, a) > 0$ s.t. if $I = [a, b]$ is an interval of length t (t is large) and $\alpha = \{J_1, \cdots, J_p\}$ is a partition of I into black and white intervals s.t.*

(1) $d(J_i, J_j) \geqq C[\min\{l(J_i), l(J_j)\}]^{1+\gamma}$ *for any two black* $J_i, J_j \in \alpha$,

(2) $l(J) \leqq ta$ *for any black* $J \in \alpha$,

(3) $l(J) \geqq r$ *for any white* $J \in \alpha$,

then $m_w(t, \alpha) \geqq \theta$ where $m_w(t, \alpha)$ denotes the total relative measure of white intervals of α on I.

PROOF. (R. Solovay) Denote $A_n = \{J \in \alpha / J$ is black and $l(J) \geqq 2^n\}$. Let $\alpha \geqq \alpha_0 \geqq \cdots \geqq \alpha_n \geqq \cdots$ be a descending sequence of partitions of I s.t. $J \in \alpha_n$ is black iff $J \in A_n$. Denote $m_n = m_w(t, \alpha_n)$. We have $m_w(t, \alpha) \leqq m_0 \leqq m_1 \leqq \cdots$.

(i) Suppose $A_0 = \emptyset$, i.e. $l(J) < 1$ for all black $J \in \alpha$. Let $I = I_w \cup I_b$ where $I_w = \bigcup_{J \in \alpha \text{ is white}} J$, $I_b = \bigcup_{J \in \alpha \text{ is black}} J$ and let k be the number of black intervals in α. If $k > 1$ there are at least $(k-1)$ white intervals whose length is at least r by (3). We have

$$\begin{aligned} m_w(t, \alpha) &= l(I_w)/[l(I_w) + l(L_b)] \\ &= \left[1 + \frac{l(L_b)}{l(I_w)}\right]^{-1} \\ &\geqq \left[1 + \frac{k}{(k-1)r}\right]^{-1} \\ &\geqq [1 + 2/r]^{-1}. \end{aligned} \tag{6}$$

If $k = 1$ then $m_w(t, \alpha) > [1 + 1/r]^{-1}$.

(ii) Suppose $A_0 \neq \emptyset$. We get as in (i)

$$m_w(t, \alpha) \geqq [1 + 1/r]^{-1} m_0. \tag{7}$$

Let $A_{n+1} \neq \emptyset$ and let D be a white interval between two consecutive black intervals of α_{n+1}. Consider α_n on D. Let $D = D_w \cup D_b$ as above. If $J \in \alpha_n \mid D$ is black, then $2^n \leqq l(J) < 2^{n+1}$.

We have by (1)

$$l(D_w) \geqq CK_n \cdot 2^{n(1+\gamma)}$$

where K_n is the number of black intervals in α_n / D.

We have

$$\begin{aligned} m_w(\alpha_n, D) &= l(D_w)/[l(D_w) + l(D_b)] \\ &= \left[1 + \frac{l(D_b)}{l(D_w)}\right]^{-1} \\ &\geqq \left[1 + \frac{K_n \cdot 2^{n+1}}{CK_n 2^{n(1+\gamma)}}\right]^{-1} \\ &= [1 + 2C^{-1} 2^{-n\gamma}]^{-1}. \end{aligned} \tag{8}$$

(8) implies $m_n \geqq [1 + 2C^{-1} 2^{-n\gamma}]^{-1} m_{n+1}$ and

(9) $$m_0 \geqq \prod_{k=0}^{N-1} [1+2C^{-1}2^{-k\gamma}]^{-1} m_N \geqq \prod_{k=0}^{\infty} [1+2C^{-1}2^{-k\gamma}]^{-1} m_N \geqq \bar{\theta} m_N$$

where $0<\bar{\theta}<\infty$ and N is s.t. $A_N \neq \varnothing$ but $A_{N+1}=\varnothing$.

(iii) We estimate m_N. If $J \in \alpha_N$ is black then $2^N \leqq l(J) \leqq 2^{N+1}$. Let $\{B_1, \cdots, B_{k(N)}\}$ be black intervals of α_N, B_j is to the right of B_i if $j>i$. Let $B_{k(N)}=[b_1,b_2]$ and $D=[a,b_1]=D_w \cup D_b$ (w.r. to α_N on D). We have

$$\begin{aligned} m_w(\alpha_N, D) &= \left[1+\frac{l(D_b)}{l(D_w)}\right]^{-1} \\ &\geqq \left[1+\frac{(k(N)-1)\cdot 2^{N+1}}{C\cdot (k(N)-1)\cdot 2^{N(1+\gamma)}}\right]^{-1} \\ &= [1+2C^{-1}2^{-N\gamma}]^{-1}. \end{aligned}$$

Since $l(B_{k(N)}) \leqq ta$ we get

(10) $$m_N \geqq [1+2C^{-1}2^{-N\gamma}]^{-1}(1-a).$$

(6), (7), (9) and (10) complete the proof. □

Let $I(u)=[u,u_t]$, $I(v)=[v,v_t]$ be two intervals, $l(I(u))=l(I(v))=t$. Let $J_1(u)<J_2(u)<\cdots<J_k(u)$ and $J_1(v)<J_2(v)<\cdots<J_k(v)$ be black subintervals of $I(u)$ and $I(v)$ respectively. We match $J_i(u)$ with $J_i(v)$ to get a match σ. Blocks of σ are pairs $B_i=\{J_i(u),J_i(v)\}$, $i=\overline{1,k}$. We define $l(B_p)=\max\{l(J_p(u)), l(J_p(v))\}$, $d(B_p,B_q)=\max\{l[z_p(u),y_q(u)],\ l[z_p(v),y_q(v)]\}$ where $J_p=[y_p,z_p]$, $p<q$, and

(11) $$\bar{f}(\sigma)=1-\frac{1}{2t}\, l\left(\bigcup_{i=1}^{k} J_i(u) \cup \bigcup_{i=1}^{k} J_i(v)\right).$$

Lemma 3. *Suppose*

(1) $d(B_p,B_q) \geqq \max\{r, C[\min\{l(B_p), l(B_q)\}]^{1+\gamma}\}$ *for some* C, γ, $r>0$, $p,q=\overline{1,k}$,

(2) $l(B_p) \leqq ta$ *for some* $0<a<1$, $p=\overline{1,k}$.

If t is sufficiently large, then $\bar{f}(\sigma) \geqq \theta/2=\tilde{\theta}>0$ *where* $\theta=\theta(C,\gamma,r,a)$ *is as in Lemma* 2.

Proof. Let $Q_i \in B_i$ be s.t. $l(Q_i)=l(B_i)$, $i=\overline{1,k}$ and $R_i=[z_i(w), y_{i+1}(w)]$ where $w=u$ if $l[z_i(u),y_{i+1}(u)]=d(B_i,B_{i+1})$ and $w=v$ otherwise, $i=\overline{0,k}$, $B_0=\{[u],[v]\}$, $B_{k+1}=\{[u_t],[v_t]\}$.

We have $I(u)\cup I(v)=\bigcup_{i=1}^{k} Q_i \cup \bigcup_{i=0}^{k} R_i \cup D$ where $l(D) \leqq t$.

Form an interval V by gluing black $\{Q_i\}$ and white $\{R_i\}$ as follows:

$$V = R_0 \cup Q_1 \cup R_1 \cup Q_2 \cup R_2 \cup \cdots \cup Q_{k-1} \cup R_{k-1} \cup Q_k \cup R_k. \tag{12}$$

It is clear that $t \leqq l(V) \leqq 2t$. This partition of V into black and white intervals satisfies the conditions of Lemma 2. By this lemma

$$m_w(V) = l\left(\bigcup_{i=0}^{k} R_i\right)\Big/ l(V) \geqq \theta > 0.$$

We have

$$\bar{f}(\sigma) \geqq l\left(\bigcup_{i=0}^{k} R_i\right)/2t \geqq l\left(\bigcup_{i=0}^{k} R_i\right)/2l(V) \geqq \frac{\theta}{2} > 0. \qquad \square$$

2. u-cylindric partitions (see [11])

Let $p\colon G \to M = \Gamma/G$ be the projection $pg = \Gamma g$, $x \in M$ and $g \in p^{-1}(x)$.

$$\tilde{W}^s(g) = \left\{ g \begin{pmatrix} e^u & v \\ 0 & e^{-u} \end{pmatrix} \middle| \, u, v \in R \right\} \qquad \text{and} \qquad W^s(x) = p\tilde{W}^s(g)$$

are leaves of stable foliations $\tilde{W}^s$ and W^s in G and in M respectively (stability w.r. to the geodesic flow). We define

$$\tilde{W}^{uu}(g) = \left\{ g \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \middle| \, u \in R \right\}, \qquad W^{uu}(x) = p\tilde{W}^{uu}(g)$$

to get strong unstable foliations in G and in M. Leaves of W^{uu} are orbits of h_t. h_t is covered by the flow H_t:

$$H_t(g) = g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

in G and leaves of $\tilde{W}^{uu}$ are orbits of H_t.

For a foliation W, $A \subset W$ means A is a subset of a leaf of W.

Definition 2 (see [11]). Two sets $A, B \subset W^s$ are called u-isomorphic ($A \overset{u}{\sim} B$) if there is a continuous $\psi\colon A \times I \to M$ ($I = [0,1]$) s.t. (1) $\psi(x, I) \subset W^{uu}$, $x \in A$, (2) $\psi(x, 0) = x$, $\psi(x, 1) \in B$ and the map $\tilde{\psi}\colon A \to B$ $\tilde{\psi}(x) = \psi(x, 1)$ is a homeomorphism. The set $\psi(A \times I) = P$ is called a u-cylinder with faces A, B. If the positive direction on orbits of h_t goes from A to B we write $A = A_1(P)$, $B = A_2(P)$. $\psi(x, I)$ and $\psi(y, I)$, $x, y \in A$ are called s-isomorphic ($\psi(x, I) \overset{s}{\sim} \psi(y, I)$).

Denote η_P the partition of P into sets u-isomorphic to $A_1 = A_1(P)$ and $\xi_P = \{\psi(x, I) \subset P, x \in A_1\}$ the partition of P into s-isomorphic intervals of orbits of h_t.

The h_t-invariant measure μ in M has the form $d\mu = d\mu' \times dt$ where μ' is a measure on leaves of W^s equivalent to the Riemannian volume on W^s and invariant under the u-isomorphism (see [3]).

Henceforth we suppose that $A_1 = \overline{\operatorname{Int} A_1} \subset W^s$ is compact and that μ'-measure of the boundary ∂A_1 of A_1 is zero.

Let $\omega = \{P_1, \cdots, P_m\}$ be a partition of M into u-cylinders (see [11]). We say ω is a u-partition. Denote $\eta_\omega = \{A \in \eta_P, P \in \omega\}$, $\xi_\omega = \{C \in \xi_P / P \in \omega\}$. We shall assume that the diameters of $A \in \eta_\omega$ in W^s and the diameters of $P \in \omega$ are at most ε for a sufficiently small $\varepsilon > 0$.

Let $Y = M \times M$, $\nu = \mu \times \mu$, $S_t = h_t \times h_t$, $\alpha = \{P \times Q \mid P, Q \in \omega\}$, $\beta = \{A_1(P) \times Q,\ P \times A_1(Q)/P,\ Q \in \omega\}$. Let $X = \bigcup \beta$ be the set-theoretic union of atoms of β. For $x \in X$ we denote $\phi(x)$ the first intersection of the orbit $\{S_t x, t > 0\}$ with X and $F(x)$ the length of $[x, \phi x]$ on the orbit. (Y, ν, S_t) is a special flow built over (X, m, ϕ) with F where m is a ϕ-invariant measure on X generated by ν.

We shall show that the partiton α does not satisfy the condition in the conclusion of Lemma 1. This will imply by the lemma that S_t is not LB.

Let T_t be the flow in $G \times G$ covering S_t, $pT_t = S_t p$ (we denote the projections $G \to \Gamma/G = M$ and $G \times G \to \Gamma \times \Gamma / G \times G = Y$ by the same symbol p).

We lift ω into G to get a u-partition Ω, $p(\Omega) = \omega$ and $\Delta = \Omega \times \Omega$ in $G \times G$, $p(\Delta) = \alpha$.

So we have (h_t, ω) in M, (H_t, Ω) in G, $p(H_t, \Omega) = (h_t, \omega)$, (S_t, α) in $M \times M = Y$ and (T_t, Δ) in $G \times G$, $p(T_t, \Delta) = (S_t, \alpha)$.

For $u \in M$, G, Y, $G \times G$, the symbol $I_t(u)$, $t > 0$ denotes the orbit interval from u to $h_t u$, $H_t u$, $S_t u$, $T_t u$ respectively.

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in G \times G$ have equal Δ-names, i.e. $y_i \in \Omega(x_i)$, $i = \overline{1,2}$. Denote $z_i = \xi_\Omega(y_i) \cap \eta_\Omega(x_i) \in \tilde{W}^s(x_i)$. We have

$$d_s(x_i, z_i) < \varepsilon, \qquad i = 1,2.$$

where d_s denotes the metric in $\tilde{W}^s$ generated by the Riemannian metric in G.

Now let x with y and $T_t x$ with $T_q y$ have equal Δ-names, $t, q \geqq 0$. This means that $H_q y_i \in \Omega(H_t x_i)$, $i = \overline{1,2}$. Denote $x_i(t) = H_t x_i$.

Let $z_1' = \xi_\Omega(y_i(q)) \cap \eta_\Omega(x_i(t)) \in \tilde{W}^s(x_i(t))$. We have $[z_i, z_i'] \overset{s}{\sim} [x_i, x_i(t)]$ and

$$d_s(x_i(t), z_i') < \varepsilon. \tag{13}$$

Let $q_i = q_i(t)$ be s.t. $z_i' = z_i(q_i) = H_{q_i} z_i$. We have

$$|q_i - q| < 2\varepsilon, \qquad i = \overline{1,2},$$

(14)
$$|q_1 - q_2| < 4\varepsilon.$$

Let $z \in G$ be s.t. $z_1 = x_1 \cdot z$ (the product in G) and let $w = w(x_1, y_1, x_2) = x_2 \cdot z \in \tilde{W}^s(x_2)$. Since the metric in G is left invariant we have

$$[x_2, x_2(t)] \overset{s}{\sim} [w, w(q_1(t))] \overset{s}{\sim} [z_2, z_2(q_2(t))].$$

Denote $q_1(t) = s(t, x_1, z_1) = s(t, x_2, w) = s$ and $q_2(t) = l(s, w, z_2) = l$.

(14) says that z_2 must be so close to w that

(15)
$$|s - l| < 4\varepsilon.$$

Clearly

(16)
$$d_s(w(s), z_2(l)) < 2\varepsilon.$$

We now make precise the closeness of z_i to x_i and of z_2 to w to satisfy (13) and (15).

Let G_t:

$$G_t g = g \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$$

and $g_t = p(G_t)$ be the geodesic flows on G and on M respectively. Let H^*_t:

$$H^*_t g = g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and $h^*_t = p(H^*_t)$. Let $\tilde{W}^{ss}$ be the foliation of G into orbits of H^*_t and $W^{ss} = p\tilde{W}^{ss}$ in M. We have

(17)
$$\begin{aligned} g_t \circ h^*_s &= h^*_{\lambda^{-t}s} \circ g_t \\ g_t \circ h_s &= h_{\lambda^t s} \circ g_t \end{aligned} \qquad \text{where } \lambda = e^2.$$

Let $z = H^*_b G_a e$ and let $[e, H_s e] \overset{s}{\sim} [z, H_{l(s)} z]$. Let $H_{l(s)} z = H^*_{b(s)} G_{a(s)} H_s e$. We get the following equations for $l(s)$, $a(s)$, $b(s)$:

$$H^*_{b(s)} G_{a(s)} H_s e = H_{l(s)} H^*_b G_a e$$

or

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} e^{a(s)} & \\ & e^{-a(s)} \end{pmatrix} \begin{pmatrix} 1 & b(s) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^a & \\ & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l(s) & 1 \end{pmatrix}.$$

One can compute that

$$a(s) = -\log(e^{-a} - sbe^{a}),$$

$$b(s) = be^{a}(e^{-a} - sbe^{a}), \tag{18}$$
$$l(s) = e^{a}s/(e^{-a} - sbe^{a}).$$

Now let $z_i' \in \tilde{W}^s(x_i(t))$ be as in (13) and let $z_i \in \tilde{W}^s(x_i)$ be $z_i = H^*_{r_i}G_{p_i}x_i$ for some $|r_i|, |p_i| < \varepsilon$, $i = \overline{1,2}$.

(13) implies by virtue of (18) that

$$|r_i| < A(\varepsilon)/t, \qquad t > 0, \quad i = \overline{1,2}, \quad \text{for some } \lim_{\varepsilon \to 0} A(\varepsilon) = 0. \tag{19}$$

We are now interested in (15). Since $z_2 \in \tilde{W}^s(w)$ we have $z_2 = H^*_b G_a w$ for some $|a|, |b| < \varepsilon$. By (16) and (18) we have

$$|b| < A(2\varepsilon)/s, \qquad s > 0$$

and

$$\begin{aligned} l - s &= l(s) - s \\ &= (e^{a}s - e^{-a}s + s^2be^{a})/(e^{-a} - sbe^{a}) \\ &= (s^2bD + Las)/(e^{-a} - sbe^{a}) \end{aligned} \tag{20}$$

where $D = e^{a}$,

$$e^{a} - e^{-a} = L(a)a = La, \qquad L_1 \leqq L \leqq L_2 \quad \text{for some } L_1, L_2 > 0.$$

Let $y \in \Delta(x)$. We are interested in all possible $q, t \geqq 0$ s.t. $T_q y \in \Delta(T_t x)$.

As before let $z_1 \in H^*_r G_p x_1$ (then $w = H^*_r G_p x_2$) and $z_2 = H^*_b G_a w$, $|p|, |r|, |a|, |b| < \varepsilon$.

Denote $O(g; \alpha, \beta, \gamma) = \{H_c H^*_b G_a g / |a| < \alpha, \ |b| < \beta, \ |c| < \gamma\}$, $\alpha, \beta, \gamma > 0$, $g \in G$ and $V(x; P, R, A, B) = \{y \in G \times G / y_i \in O(x_i; P, R, 8\varepsilon), \ i = \overline{1,2}$ and $y_2 \in O(w; A, B, 8\varepsilon)\}$, $P, R, A, B > 0$.

It follows from (15), (16), (19) that we are actually interested in all possible $t \geqq 0$ s.t. if we denoe $q_1(t) = s(t, x_2, w) = s(t, p, r) = s$ and $q_2(t) = l = l(s, w, z_2) = l(s, a, b)$, $l(0, a, b) = 0$ then $|r| < A(\varepsilon)/t$, $|b| < A(2\varepsilon)/s$ and $|l - s| < 4\varepsilon$.

Denote $J(m) = \{s \in R^+ / s \leqq A(2\varepsilon)/|b|, \ |l - s| \leqq m\varepsilon\}$ and $Q(m) = \{s \in R^+ / s \leqq A(2\varepsilon)/|b|, |s^2bD + Las| \leqq m\varepsilon\}$. We have from (20)

$$J(m) \subset Q(2m) \subset J(4m) \qquad \text{if } \varepsilon \text{ is small enough.}$$

In particular, $J(4) \subset Q(8) \subset J(16)$.

Let $P(x, y) = \{t \in R^+ / t \leqq A(\varepsilon)/|r|, \ s \leqq A(2\varepsilon)/|b|, |s^2bD + Las| < 8\varepsilon\}$.

Let $\bar{P} = \bar{P}(x, y) = [0, \tau]$ be the connected component of $P(x, y)$ containing zero.

Take $y_1^\tau \in \tilde{W}^s(x_1(\tau))$ and let $y_1^\tau = H_{\beta(\tau)}y_1$.

We denote $B(x,y) = \{I_\tau(x), I_{\beta(\tau)}(y)\}$ and call $B = B(x,y)$ the black block of (x,y), $y \in \Delta(x)$.

We say that $(\bar{x},\bar{y}) \in B(x,y)$ if $\bar{x} = T_t x$ for some $0 \leqq t \leqq \tau$ and $\bar{y}_1 \in 0(y_1^t, 0, 0, 60\varepsilon)$. In particular, $(T_t x, T_{\beta(t)}y) \in B(x,y)$, $0 \leqq t \leqq \tau$.

It follows from (20) that

$$|\beta(\tau) - \tau| < E(\varepsilon)\tau \qquad \text{for some } 0 < E(\varepsilon) \to 0, \quad \varepsilon \to 0$$

and so the length

$$l(B) = \max\{\tau, \beta(\tau)\} < \tau(1 + E(\varepsilon)) = D_1\tau. \tag{21}$$

Assume that

$$\max\{|r|,|a|,|b|\} \neq 0. \tag{22}$$

There are the following possibilities:

(1) $|a| = |b| = 0$. Then $|r| \neq 0$ and $\bar{P} = [0,\tau]$ with $\tau = A(\varepsilon)/|r|$. So $l(B(x,y)) < D_1 A(\varepsilon)/|r|$ or

$$|r| < D_1 A(\varepsilon)/l(B). \tag{23}$$

Let $t \in [0,\tau]$ and let $y_1^t = H^*_{r(t)}G_{p(t)}x_1(t)$. It follows from (18) and (23) that $|p(t)| < C_1(\varepsilon)$ and $|r(t)| < C_2(\varepsilon)/l(B)$ where $0 < C_1(\varepsilon)$, $C_2(\varepsilon) \to 0$, $\varepsilon \to 0$.

Since $z_2^t = w^t$, $t \in \bar{P}$, this says that

$$\text{if } (\bar{x},\bar{y}) \in B(x,y) \text{ then } \bar{y} \in V(\bar{x}; C_1(\varepsilon), C_2(\varepsilon)/l(B), 0, 0). \tag{24}$$

(2) $\max\{|a|,|b|\} \neq 0$. Denote $Q = \{s \in R \mid |s^2 bD + Las| < 8\varepsilon\}$.

Let (i) $(La)^2 \leqq 32\varepsilon\,|bD|$.

Then $b \neq 0$ and $Q = Q_0 \ni 0$ is an interval of length

$$l(Q_0) = \sqrt{(La)^2 + 32\varepsilon\,|bD|}/|bD| \leqq K_1/\sqrt{|b|}$$

by (i) and (20) for some $K_1 > 0$ depending only on ε.

Using (i) and (20) we get

$$\sqrt{|b|} \leqq K_1/l(Q_0), \quad |a| \leqq K_2/l(Q_0), \quad K_2 > 0. \tag{25}$$

Let (ii) $(La)^2 > 32\varepsilon\,|bD|$.

Then $a \neq 0$. If $b = 0$ then $Q = Q_0 \ni 0$ is an interval of length

(26) $$l(Q_0)=16\varepsilon/|La|.$$

If $b\neq 0$ then $Q=Q_0\cup Q_1$, $0\in Q_0$, and

$$l(Q_0)=l(Q_1)=|(\sqrt{(La)^2+32\varepsilon|bD|}-\sqrt{(La)^2-32\varepsilon|bD|})/2bD|.$$

If $32|bD|\leqq(La)^2$ then $32\varepsilon|bD|/(La)^2\leqq\varepsilon$ and we get

(27) $$l(Q_0)=l(Q_1)\leqq K_3/|La| \quad \text{if } \varepsilon \text{ is sufficiently small,} \qquad K_3>0.$$

If $32|bD|>(La)^2$ or $|bD|>(La)^2/32$ then

(28) $$l(Q_0)=l(Q_1)\leqq K_4/|La|, \qquad K_4>0.$$

We have from (ii), (25), (26), (27), (28) that if $\max\{|a|,|b|\}\neq 0$ then

(29) $$|a|\leqq K_5/l(Q_0), \quad \sqrt{|b|}\leqq K_6/l(Q_0), \qquad K_5, K_6>0.$$

Let $Q=Q_0\cup Q_1$, where Q_0 is the connected component of Q, containing 0 and Q_1 might be empty.

Let $0\leqq s\in Q$ and $z_2^s\in\tilde{W}^s(w(s))$, $z_2^s=H^*_{b(s)}G_{a(s)}w(s)$.

Let $s\in Q_0$, then $s\leqq l(Q_0)$ and it follows from (29) and (18) that

(30) $$|a(s)|, \sqrt{|b(s)|}<K_7/l(Q_0) \quad \text{if } l(Q_0) \text{ is large enough,} \qquad K_7>0.$$

Let $0<s\in Q_1$ be s.t. $s\leqq A(2\varepsilon)/|b|$. Then $|l-s|<16\varepsilon$ and we have using (18)

$$e^{-a}\left(1-\frac{16\varepsilon}{s}\right)<(e^{-a}-sbe^{a})^{-1}<e^{-a}\left(1+\frac{16\varepsilon}{s}\right).$$

This implies by (29)

(31) $$|a(s)|, \sqrt{|b(s)|}<K_8/l(Q_0), \quad K_8>0, \qquad \text{since } s\geqq l(Q_0).$$

We have $\bar{P}=P(x,y)\cap\{t\in R^+/s(t)\in Q_0\}$ and $l(B(x,y))\leqq D_1 l(Q_0)$ by (21).

Let $t_2>t_1>\tau$, $q_2>q_1>0$ be s.t.

(32) $$T_{q_i}y\in\Delta(T_{t_i}x), \qquad i=1,2.$$

Then $s(t_i)\leqq A(2\varepsilon)/|b|$, $s(t_i)\in Q_1$ and $s(t_2)-s(t_1)\leqq l(Q_1)=l(Q_0)$.

Denote $B'=\{J(x),J(y)\}$ where $J(x)$ is the orbit interval $[T_{t_1}x, T_{t_2}x]$ and $J(y)$ is the orbit interval $[T_{q_1}y, T_{q_2}y]$.

We get the following consequence from (24), (30), (31), and (32).

CONSEQUENCE. Let (22) hold. Then there are $C(\varepsilon)$, $K > 0$ depending only on $\varepsilon > 0$, $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$ s.t. if we denote $V(x; l) = V(x; C(\varepsilon), R, A, B^2)$, $R = \min\{C(\varepsilon), C(\varepsilon)/l\}$, $A = \min\{C(\varepsilon), K/l\}$, $B^2 = \min\{C(\varepsilon), [K/l]^2\}$, $l > 0$ then the following holds:

(33)
(1) If $(\bar{x}, \bar{y}) \in B(x, y)$ then $\bar{y} \in V(\bar{x}; l(B))$.
(2) If $(\bar{x}, \bar{y}) \in B$, $(\bar{x}', \bar{y}') \in B'$ then $\bar{y} \in V(\bar{x}; \max\{l(B), l(B')\})$ and $\bar{y}' \in V(\bar{x}'; \max\{l(B), l(B')\})$.

Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in Y = M \times M$, $v \in \alpha(u)$. Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in G \times G$, $px_i = u_i$, $py_i = v_i$, $d(x_i, y_i) < \varepsilon$, $y \in \Delta(x)$. Let $B(x, y)$ be the black block of (x, y). We call $pB(x, y) = B(u, v)$ the black block of (u, v) in Y. This definition does not depend on $x \in p^{-1}(u)$, $y \in p^{-1}(v)$ if $y \in \Delta(x)$ and ε is sufficiently small.

Let $u' \in \{S_t u, t \geqq 0\}$, $v' \in \{S_t v, t \geqq 0\}$, $x' \in \{T_t \dot{x}, t \geqq 0\}$, $y' \in \{T_t y, t \geqq 0\}$, $px' = u'$, $py' = v'$. We say that $(u', v') \in B(u, v)$ iff $(x', y') \in B(x, y)$. We say that $v' \in V(u', l)$ iff $y' \in V(x', l)$.

Suppose that $S_q v \in \alpha(S_t u)$ for some $q, t \geqq 0$. We have $p(T_t x) = S_t u$ and $p(T_q y) = S_q v$.

It is not necessarily true that $T_q y \in \Delta(T_t x)$. But there is a unique $(A_1, A_2) \in \Gamma \times \Gamma \subset G \times G$ s.t.

$$(A_1 \cdot H_q y_1, A_2 \cdot H_q y_2) \in \Delta(T_t x) \tag{34}$$

(remember that $(\Delta, T_t = H_t \times H_t)$ in $G \times G$ cover $(\alpha, S_t = h_t \times h_t)$ in $M \times M = Y$).

We write $(u, v) \xrightarrow{\Gamma} (S_t u, S_q v)$ if $(A_1, A_2) \neq (e, e)$ in (34) and $(u, v) \xrightarrow{e} (S_t u, S_q v)$ if $(A_1, A_2) = (e, e)$ in (34).

It is clear that $(u, v) \xrightarrow{e} (S_t u, S_q v)$ iff $T_q \in \Delta(T_t x)$.

Let $O_{C(\varepsilon)}(x)$ denote the $C(\varepsilon)$-ball in $G \times G$ centered at $x \in G \times G$ ($C(\varepsilon)$ is as in (33)).

If ε is sufficiently small then there is $r > 0$ depending only on Γ s.t. if $y \in O_{C(\varepsilon)}(x)$ and $(A_1 \cdot H_q y_1, A_2 \cdot H_q y_2) \in O_{C(\varepsilon)}(T_t(x))$ for some $t, q \in R$, $(A_1, A_2) \neq (e, e)$ then

$$\max\{|t|, |q|\} > r. \tag{35}$$

In the next paragraph we shall prove the following lemma.

LEMMA 4. *There are $C, \gamma' > 0$ depending only on Γ and ε with the following property: given $\rho > 0$ there are $l_0 = l_0(\rho) > 0$ and closed set $F = F(\rho) \subset Y$,*

$\nu(F) > 1 - \rho$ *s.t. if* $u \in F$, $l \geqq l_0$, $v \in V(u;l)$, $S_q v \in V(S_t u;l)$, $t, q \geqq 0$ *and* $(u,v) \overset{\mathcal{E}}{\sim} (S_t u, S_q v)$ *then* $\max\{t,q\} > Cl^{1+\gamma'}$.

We are going to prove Theorem 1 using (33) and Lemmas 1, 3 and 4.

Let $u, v \in Y$ and let σ be an α-match between $I_t(u)$, $I_t(v)$, $t > 0$. Let $B_1 < B_2 < \cdots < B_m$ be α-blocks of σ, $B_i \in \sigma$, $i = \overline{1,m}$ (see paragraph 1 for definitions). We write $B_i < B_{i+1}$ if B_{i+1} is on the right of B_i on orbits $I_t(u)$, $I_t(v)$.

We now construct a black extension $\bar{\sigma}$ of σ by the following procedure.

Let $B_i = \{[u_i, u_i'], [v_i, v_i']\}$, $[u_i, u_i'] \subset I_t(u)$, $[v_i, v_i'] \subset I_t(v)$, $i = \overline{1,m}$.

Take (u_1, v_1) and let $B(u_1, v_1)$ denote the black block of (u_1, v_1). Let $i_1 \in \{1, 2, \cdots, m\}$ be s.t. $(u_{i_1}, v_{i_1}) \in B(u_1, v_1) = \{J(u_1), J(v_1)\}$ and $(u_k, v_k) \not\in B(u_1, v_1)$ if $k > i_1$.† It follows from the definition of black blocks that if $(u_{i_1}, v_{i_1}) \in B(u_1, v_1)$ then B_{i_1} is a subblock of $B(u_1, v_1)$, i.e. $[u_{i_1}, u_{i_1}'] \subset J(u_1)$, $[v_{i_1}, v_{i_1}'] \subset J(v_1)$.

Denote $\bar{B}_1 = \{[u_1, u_{i1}'], [v_1, v_{i1}']\}$.

Suppose we constructed $\bar{B}_j = \{[u_{i_{j-1}+1}, u_{i_j}'], [v_{i_{j-1}+1}, v_{i_j}']\}$, $i_0 = 0$, $i_j \geqq i_{j-1} + 1$, $j = 1, \ldots, n$.

To find i_{n+1} we take (u_{i_n+1}, v_{i_n+1}) and consider the following cases.

(1) There is $1 \leqq k \leqq n$ s.t. $(u_{i_{k-1}+1}, v_{i_{k-1}+1}) \overset{\mathcal{E}}{\sim} (u_{i_n+1}, v_{i_n+1})$. Then i_{n+1} is the largest $m \geqq i_n + 1$ s.t. $(u_m, v_m) \overset{\mathcal{E}}{\sim} (u_{i_{k-1}+1}, v_{i_{k-1}+1})$ (see (33)).

(2) There is no k as in (1). Then we define i_{n+1} in the same way as we did i_1.

So we set $\bar{\sigma} = \{\bar{B}_1, \cdots, \bar{B}_{\bar{m}}, \bar{m} \leqq m\}$.

It follows from (33) that if $\bar{B}_i \doteq \{[\bar{u}_i, \bar{u}_i'], [\bar{v}_i, \bar{v}_i']\} \in \bar{\sigma}$ then $\bar{v}_i \in V(\bar{u}_i, l(\bar{B}_i))$, $\bar{v}_i' \in V(\bar{u}_i', l(\bar{B}_i))$ and if $(\bar{u}_i, \bar{v}_i) \overset{\mathcal{E}}{\sim} (\bar{u}_k, \bar{v}_k)$ (there are at most two such B_i, B_k) then B_i and B_k satisfy (2) of (33).

Proof of Theorem 1. We shall prove the following assertion. There exists $\delta > 0$ with the following property: given $\beta > 0$ there exists $t_0 = t_0(\beta) > 0$ and a set $Y(\beta) \subset Y$, $\nu(Y(\beta)) > 1 - \beta$ s.t. if $u \in Y(\beta)$, $t \geqq t_0$ then there is a set $Z(u;t) \subset Y$, $\nu(Z(u,t)) > 1 - \beta$ s.t. if $v \in Z(u,t)$ then $\bar{f}(I_t(u), I_t(v)) \geqq \delta$ (see paragraph 1 for the definition of $\bar{f}$).

Clearly, this assertion contradicts the conclusion of Lemma 1. By this lemma S_t is not LB.

Let $C, \gamma' > 0$ be as in Lemma 4. Let $\bar{l} > 0$ and $0 < \gamma < \min\{\gamma', 1/15\}$ be s.t. if $l \geqq \bar{l}$ then

$$(36) \qquad Cl^{1+\gamma'} > (3l^{1+\gamma})^{1+\gamma} \quad \text{and} \quad K/l < C(\varepsilon) \quad \text{in (33).}$$

† In other words, i_1 is the largest i s.t. $u_i \in J(u_1)$ and $(u_i, v_i) \overset{\mathcal{E}}{\sim} (u_1, v_1)$.

Let $r>0$ be defined by (35).

Let $\tilde{\theta}=\tilde{\theta}(C,\gamma,r,\frac{1}{4})$ be as in Lemma 3.

Let $\rho=\tilde{\theta}/100$ and let $l_0=l_0(\rho)>0$ and $F=F(\rho)$, $\nu(F)>1-\tilde{\theta}/100$ be as in Lemma 4.

Denote $\bar{Z}=\bar{Z}(t,u)=\bigcup_{s=0}^{t}\bigcup_{p=0}^{t}S_{-s}V(S_pu,t^{3/4})$. It follows from the definition of $V(u;l)$ that $\nu V(u;l)<Ll^{-4}$ for some $L>0$ depending only on ε, $u\in Y$. We get

$$\nu(Z(u,t))>1-Lt^{-1}.$$

If $v\in Z(u,t)$ and $S_qv\in\alpha(S_pu)$, $0\leqq p,\ q\leqq t$ then (22) holds for S_pu, S_qv and we may use (33). (33) says that the lengths of black blocks on $I_t(u)$, $I_t(v)$ are at most $t^{3/4}$.

Let $\beta>0$ be fixed. Let $\tilde{t}>0$ be s.t. if $t>\tilde{t}$ then

$$\nu(Z(t,u))>1-\beta,\qquad u\in Y.$$

(37) Since S_t is ergodic there is $t_0=t_0(\beta)>\tilde{t}$ and a set $Y(\beta)\subset Y$, $\nu(Y(\beta))>1-\beta$ s.t. if $u\in Y(\beta)$, $t>t_0$ then the relative Lebesque measure of F on $I_t(u)$ is at least $1-2\rho=1-\tilde{\theta}/50$.

Let $u\in Y(\beta)$ and $v\in Z(u,t)$, $t>t_0>\tilde{t}>0$. Let σ be an α-match between $I_t(u)$, $I_t(v)$ and let $\bar{\sigma}=\{\bar{B}_i,\ i=1,\cdots,\bar{m}\}$ be the black extension of $\sigma,\bar{B}_i=\{[\bar{u}_i,\bar{u}_i'],[\bar{v}_i,\bar{v}_i']\}$. We have

$$\bar{f}(\sigma)\geqq\bar{f}(\bar{\sigma})\quad\text{and}\quad l(\bar{B})<t^{3/4},\qquad \bar{B}\in\bar{\sigma}. \tag{38}$$

It follows from the definitions of r in (35) that

(39) if $(\bar{u}_i',\bar{v}_i')\overset{\mathcal{L}}{\sim}(\bar{u}_j,\bar{v}_j)$, $j>i$ then $d(\bar{B}_i,\bar{B}_j)>r$ (see paragraph 1 for the definition of d).

Let $\bar{l}=\max\{l_0,\tilde{l},r\}$ and let $\bar{\sigma}'=\{\bar{B}\in\bar{\sigma}/l(\bar{B})\geqq\bar{l}\}=\{\bar{B}_1',\cdots,\bar{B}_{\bar{m}'}'\}$, $\bar{B}_1'=\{J_i'(u),J_i'(v)\}$.

Let $\tilde{\sigma}=\{\bar{B}_i'\in\bar{\sigma}'/J_i'(u)\cap F\neq\varnothing\}=\{\tilde{B}_1,\cdots,\tilde{B}_{\tilde{m}},\quad \tilde{m}\leqq\bar{m}'\leqq\bar{m}\leqq m\}$, $\tilde{B}_i=\{[\tilde{u}_i,\tilde{u}_i'],[\tilde{v}_i,\tilde{v}_i']\}$.

Let $\hat{u}_i,\hat{u}_i'\in[\tilde{u}_i,\tilde{u}_i']\cap F$ be s.t. $(\tilde{u}_i,\hat{u}_i)\cap F=\varnothing=(\hat{u}_i',\tilde{u}_i')$.

Let $\hat{v}_i,\hat{v}_i'$ be s.t. $(\hat{u}_i,\hat{v}_i)$, $(\hat{u}_i',\hat{v}_i')\in\tilde{B}_i$.

Denote $\hat{B}_i=\{[\hat{u}_i,\hat{u}_i'],[\hat{v}_i,\hat{v}_i']$ and $\hat{\sigma}=\{\hat{B}_1,\cdots,\hat{B}_m,\hat{m}\leqq\tilde{m}\}$.

It follows from (37) and (21) that

$$0\leqq\bar{f}(\hat{\sigma})-\bar{f}(\bar{\sigma}')\leqq\tilde{\theta}/50(1+E(\varepsilon))<\tilde{\theta}/25\quad\text{if }\varepsilon\text{ small enough}. \tag{40}$$

We have by (33) that

(41) $$\hat{v}_i \in V(\hat{u}_i, l(\tilde{B}_i)), \quad \hat{v}'_i \in V(\hat{u}'_i, l(\tilde{B}_i)) \quad \text{and} \quad l(\tilde{B}_i) \geqq \bar{l}.$$

It follows from the definition of r in (35) that if $(\hat{u}'_i, \hat{v}'_i) \stackrel{\mathcal{L}}{\sim} (\hat{u}_j, \hat{v}_j)$, $j > i$ then $d(\hat{B}_i, \hat{B}_j) > r$ (see also (39)).

Take $\hat{B}_1 \in \hat{\sigma}$ and consider the following possibilities:

(1) There is no $j \in \{2, \cdots, \hat{m}\}$ s.t. $(\hat{u}_1, \hat{v}_1) \stackrel{\mathcal{L}}{\sim} (\hat{u}_j, \hat{v}_j) \stackrel{\mathcal{L}}{\sim} (\hat{u}'_1, \hat{v}'_1)$. In this case we define $B^0_1 = \hat{B}_1$.

(2) There is $j_1 \in \{2, \cdots, \hat{m}\}$ s.t. $(\hat{u}'_1, \hat{v}'_1) \stackrel{\mathcal{L}}{\sim} (\hat{u}_{j_1}, \hat{v}_{j_1})$ (there is only one such j_1) and $d(\hat{B}_1, \hat{B}_{j_1}) < \max\{r, [l(\hat{B}_1)]^{1+\gamma}, [l(\hat{B}_{j_1})]^{1+\gamma}\}$. We define $B^0_1 = \{[\hat{u}_1, \hat{u}'_{j_1}], [\hat{v}_1, \hat{v}'_{j_1}]\} = \{[u^0_1, u^{0'}_1], [v^0_1, v^{0'}_1]\}$.

We have from (38) and our choice of γ in (36)

(42) $$l(B^0_1) < 3[\max\{l(\hat{B}_1), l(\hat{B}_{j_1})\}]^{1+\gamma}, \qquad 3r < 3t^{(3/4)(1+\gamma)} < 3t^{4/5} < t/4$$

if t is big enough.

We have from (33) and (41)

$$v^0_i \in V(u^0_i, l), \quad v^{0'}_i \in V(u^{0'}_i, l) \qquad \text{where } l \geqq \max\{l(\hat{B}_1), l(\hat{B}_{j_1})\},$$

(43) $$l \geqq \bar{l}, \quad l(B^0_1) \leqq 3l^{1+\gamma} \quad \text{and} \quad [l(B^0_1)]^{1+\gamma} \leqq (3l^{1+\gamma})^{1+\gamma} < Cl^{1+\gamma'}$$

by our choice of γ and $\bar{l} \leqq l$.

(3) There is $j_1 \in \{2, \cdots, \hat{m}\}$ as in (2) and

$$d(\hat{B}_1, \hat{B}_{j_1}) > \max\{r, [l(\hat{B}_1)]^{1+\gamma}, [l(\hat{B}_{j_1})]^{1+\gamma}\}.$$

In this case we set $B^0_1 = \hat{B}_1$.

Let $B^0_n = \{[\hat{u}_{j_{n-1}+1}, \hat{u}'_{j_n}], [\hat{v}_{j_{n-1}+1}, \hat{v}'_{j_n}]\}$, $j_0 = 0$. To define B^0_{n+1} we apply the above construction to $\hat{B}_{j_n+1} \in \hat{\sigma}$.

Let $\sigma^0 = \{B^0_i, i = 1, \cdots, m^0, m^0 \leqq m\}$. It follows from the construction of σ^0 that $d(B^0_i, B^0_j) > r$, $i \neq j$. This with (42) and (43) shows via Lemma 4 that σ^0 satisfies all the conditions of Lemma 3 with $a = \frac{1}{4}$. By this lemma

$$\bar{f}(\hat{\sigma}) \geqq f(\sigma^0) \geqq \tilde{\theta} = \tilde{\theta}(C, \gamma, r, \tfrac{1}{4}).$$

By (40)

(44) $$\bar{f}(\bar{\sigma}') \geqq 24\tilde{\theta}/25 = \bar{\theta}.$$

$\bar{\sigma}$ differs from $\bar{\sigma}'$ by blocks of length $< \bar{l}$.

For $\bar{B}'_i \in \bar{\sigma}'$ denote $\bar{B}'_i = D_i = \{[d_i(u), d'_i(u)],\ [d_i(v), d'_i(v)]\}$, $i = 1, \cdots, n$, $n = \bar{m}'$.

Let $\chi_i = \{E \in \bar{\sigma}/E$ is a block between D_i and $D_{i+1}\} = \{E_{i1}, \cdots, E_{ik(i)}\}$, $l(E_{ik}) < \bar{l}$, $i = \overline{1, n}$, $k = \overline{1, k(i)}$, $E_{ik} = \{[e_{ik}(u), e'_{ik}(u)],\ [e_{ik}(v), e'_{ik}(v)]\}$.

Take $E_{i1} \in \chi_i$ and consider the following possiblities.

(1) $d(D_i, E_{i1}) < r$. This means that $(d'_i(u), d'_i(v)) \overset{\varepsilon}{\sim} (e_{i1}(u), e_{i1}(v))$ and $d(E_{i1}, Q) > r$ where $Q = D_{i+1}$, E_{ik}, $k > 1$. We set

$$\bar{E}_{i1} = \{[d'_i(u), e'_{i1}(u)], [d'_i(v), e'_{i1}(v)]\}.$$

(2) There is $j_1 > 1$ s.t. $d(E_{i1}, E_{ij_1}) < r$. This means that $(e'_{i1}(u), e'_{i1}(v)) \overset{\varepsilon}{\sim} (e_{ij_1}(u), e_{ij_1}(v))$ and $d(E_{i1}, D_i) > r$ and $d(E_{ij_1}, Q) > r$ where $Q = E_{ik}$, $k > j_1$, D_{i+1}. We set

$$\bar{E}_{i1} = \{[e_{i1}(u), e'_{ij_1}(u)], [e_{i1}(v), e'_{ij_1}(v)]\}.$$

(3) $d(E_{i1}, D_{i+1}) < r$. Then $\chi_i = \{E_{i1}\}$ and $d(E_{i1}, D_i) > r$. We set

$$\bar{E}_{i1} = \{[e_{i1}(u), d_{i+1}(u)], [e_{i1}(v), d_{i+1}(v)]\}.$$

(4) $d(E_{i1}, Q) > r$ where $Q = D_i$, D_{i+1}, E_{ik}, $k > 1$. We set $\bar{E}_{i1} = E_{i1}$.

Let $\bar{E}_{ik} = \{[e_{ij_{n-1}+1}(u), e'_{ij_n}(u)], [e_{ij_{n-1}+1}(v), e'_{ij_n}(v)]\}$, $j_0 = 0$. To get $\bar{E}_{i(k+1)}$ we apply the above construction to $E_{ij_n+1} \in \chi_i$.

Denote $\bar{\chi}_i = \{\bar{E}_{i1}, \cdots, \bar{E}_{ik(i)}\}$. We have $l(\bar{E}) < 2\bar{l} + r$, $\bar{E} \in \chi_i$ and $d(\bar{E}_{ik}, \bar{E}_{ij}) > r$, $k \neq j$, $i = \overline{1, n}$. If $d(\bar{E}_{i1}, D_i) = 0$ then $d(\bar{E}_{i1}, D_{i+1}) > r$ and if $d(\bar{E}_{ik(i)}, D_{i+1}) = 0$ then $d(\bar{E}_{\overline{ik(i)}}, D_i) > r$.

Denote $\bar{\chi} = \bigcup_{i=1}^{n} \bar{\chi}_i \cup \bar{\sigma}'$. We have

$$\bar{f}(\sigma) \geqq \bar{f}(\bar{\sigma}) \geqq \bar{f}(\bar{\chi}).$$

Arguing as in the proof of Lemma 3 (see (12)) we get by using (44)

$$\bar{f}(\bar{\chi}) \geqq \bar{\theta}[2(1 + r^{-1}(2\bar{l} + r))]^{-1}.$$

We complete the proof setting $\delta = \bar{\theta}[2(1 + r^{-1}(2\bar{l} + r))]^{-1}$. □

3.1. Proof of Lemma 4. Let $u \in M$ and $v \in O(u; K/l, K^2/l, 8\varepsilon/l)$ where $K > 0$ is as in (33).

Let $0 < \gamma < 1$ be chosen later. It follows from (18) that if $0 \leqq q \leqq l^{\gamma}$ then

$$(45) \qquad h_q(v) \in O(h_q(u);\quad Q/l^{(1-2\gamma)}, Q/l, Q/l^{(1-2\gamma)}) \subset O(h_q(u); Q/l^{(1-2\gamma)})$$

for some $Q > 0$ not depending on γ if l is sufficiently big. $O(u; p)$ denotes $O(u; p, p, p)$.

Let $u \in Y$, $x = (x_1, x_2) \in G \times G$, $px = u$ and $z \in O_{C(\varepsilon)}(e) \subset G$ ($C(\varepsilon)$ is as in (33)). Denote $u(z) = p(x_1 \cdot z, x_2 \cdot z) = (u_1(z), u_2(z))$.

Let $W(u, l) = \{v \in Y / v_1 = u_1(z)$ for some $z \in O_{C(\varepsilon)}(e)$ and $v_2 \in O(u_2(z); 2Q/l)\}$, where $Q > 0$ is as in (45).

For β, $l > 0$ denote

$$E(\beta, l) = \{u \in Y/\text{there are } z \in O_{C(\varepsilon)}(e) \text{ and } 0 \leqq t, q \leqq l^\beta$$

$$\text{s.t. } S_q u(z) \in W(S_t u, l) \text{ and } (u, u(z)) \overset{\Gamma}{\sim} (S_t u, S_q u(z))\}.$$

We shall prove the following lemma.

Lemma 5. *There are β_0, θ, R, $N > 0$ depending only on Γ and ε s.t. if $0 < \beta \leqq \beta_0$, $l \geqq N$ then $\nu(E(\beta, l)) < Rl^{-\theta}$.*

Let us show how to prove Lemma 4 using Lemma 5.

Proof of Lemma 4. Let $0 < \gamma < 1$ in (45) be s.t. $\gamma/(1-2\gamma) < \beta_0$. Denote

$$Z(l) = \{u \in Y/\text{there are } v \in V(u, l) \text{ and } 0 \leqq q, t \leqq l^{1+\gamma}$$

$$\text{s.t. } S_q v \in V(S_t u, l) \text{ and } (u, v) \overset{\Gamma}{\sim} (S_t u, S_q v)\}.$$

It is clear that $Z(l)$ is open in Y.

Let $u \in Z(l)$ and $v \in V(u, l)$. It follows from the definition of $V(u; l)$ that $v_1 = u_1(z)$ for some $z \in O(e; C(\varepsilon), C(\varepsilon)/l, 8\varepsilon)$ and $v_2 \in O(u_2(z); K/l, K^2/l^2, 8\varepsilon)$.

Let $\lambda^s = l$. We have $v_1^s = g_{-s} v_1 = g_{-s} u_1(z) = u_1^s \cdot z^s$ where $z^s \in O(e; C(\varepsilon), C(\varepsilon), 8\varepsilon/l)$;

$$v_2^s = g_{-s} v_2 \in O(g_{-s} u_2(z); K/l, K^2/l, 8\varepsilon/l); \tag{46}$$

$S_q v \in V(S_t u, l)$, $0 \leqq q, t \leqq l^{1+\gamma}$ means that there is $\bar{z} \in O(e; C(\varepsilon), C(\varepsilon)/l, 8\varepsilon)$ s.t.

$$h_q v_1 = (h_t u_1) \cdot \bar{z} \quad \text{and} \quad h_q v_2 \in O((h_t u_2) \cdot \bar{z}, K/l, K^2/l^2, 8\varepsilon).$$

This implies that

$$\begin{aligned} &h_{q/l} v_1^s = h_{t/l} u_1^s \cdot \bar{z}^s, \bar{z}^s \in O(e; C(\varepsilon), C(\varepsilon), 8\varepsilon/l), \\ &h_{q/l} v_2^s \in O((h_{t/l} u_2^s) \cdot \bar{z}^s, K/l, K^2/l, 8\varepsilon/l). \end{aligned} \tag{47}$$

(45), (46) and (47) imply that

$$h_{q/l}(u_2^s \cdot z^s) \in O((h_{t/l} u_2^s) \cdot \bar{z}^s, 2Q/l^{(1-2\gamma)}).$$

Denote $u^s = (u_1^s, u_2^s)$. We get $S_{q/l}(u^s \cdot z^s) \in W(S_{t/l} u^s, l^{l(1-2\gamma)})$,

$(u^s, u^s\cdot z^s) \stackrel{\mathcal{L}}{\sim} (S_{t/l}u^s, S_{q/l}(u^s\cdot z^s))$ and $0 \leqq q/l,\ t/l \leqq l^{\gamma}$.

This means that $u^s = (g_{-s}u_1, g_{-s}u_2) = (g_{-s}\times g_{-s})(u) \in E(\gamma/(1-2\gamma), l^{(1-2\gamma)})$ or $(g_{-s}\times g_{-s})Z(l) \subset E(\gamma/(1-2\gamma), l^{(1-2\gamma)})$.

By our choice of γ: $\gamma/(1-2\gamma) < \beta_0$. We apply Lemma 5 to get $\nu(Z(l)) < Rl^{-\theta(1-2\gamma)}$ if $l \geqq N$.

Denote $\bar{Z}(l) = Y - Z(l)$.

It follows from the definition of $Z(l)$ that if $u \in \bar{Z}(l)$, $v \in V(u;l)$, $S_q v \in V(S_t u, l)$ for some $t, q \geqq 0$ and $(u,v) \stackrel{\mathcal{L}}{\sim} (S_t u, S_q v)$ then $\max\{t,q\} > l^{1+\gamma}$.

Let $\rho > 0$ be fixed. Denote $l_k = 2^{k/\theta(1-2\gamma)}$. Let $k_0 > 0$ be s.t. $l_{k_0} \geqq N$ and $R\sum_{k=k_0}^{\infty} 2^{-k} < \rho$.

Denote $F = F(\rho) = \bigcap_{k=k_0}^{\infty} \bar{Z}(l_k)$ and $l_0 = l_0(\rho) = l_{k_0}$.

We have that F is closed in Y and $\nu(F) > 1-\rho$.

Let $u \in F$, $l \geqq l_0$, $v \in V(u;l)$, $S_q v \in V(S_t u, l)$, $t, q \geqq 0$ and $(u,v) \stackrel{\mathcal{L}}{\sim} (S_t u, S_q v)$.

Let $k \geqq k_0$ be s.t. $l_k \leqq l < l_{k+1}$. $u \in F$ implies that $u \in \bar{Z}(l_k)$ and therefore

$$\begin{aligned}
\max\{t,q\} &> l_k^{1+\gamma} \\
&= 2^{k(1+\gamma)/\theta(1-2\gamma)} \\
&= 2^{k\sigma(1+\gamma)} \\
&= 2^{(k+1)\sigma(1+\gamma)}/2^{\sigma(1+\gamma)} \\
&= Cl_{k+1}^{1+\gamma} \\
&> Cl^{1+\gamma}
\end{aligned}$$

where $C = 2^{-\sigma(1+\gamma)} = 2^{-(1+\gamma)/\theta(1-2\gamma)}$.

This completes the proof. □

3.2. Proof of Lemma 5. We are going to prove Lemma 5. Let $\Gamma = \{e, -e, C_1, C_2, \cdots\}$, $|\operatorname{Tr} C_i| > 2$ and let F be the fundamental region of Γ, containing e. The closure $\bar{F}$ of F is compact. Let λ_i denote the eigenvalue of C_i with $|\lambda_i| > 1$.

Proposition 1. *There exists $\tau > 1$ s.t. $|\lambda_i| > \tau$ for all $i = 1, 2, \cdots$.*

Proof (J. Wolf). Suppose on the contrary that there is a subsequence $\{C_{i_n}, n = 1, 2, \cdots\} \subset \{C_i, i = 1, 2, \cdots\}$ s.t. $\lim_{n\to\infty} \lambda_{i_n} = 1$, $\lambda_{i_n} > 1$. (If $\lim \lambda_{i_n} = -1$ for some $\{C_{i_n}\}$ then $\lim \lambda(-eC_{i_n}) = 1$.) Let $C_{i_n} = P_n \Lambda_n P_n^{-1}$, where

$$\Lambda_n = \begin{pmatrix} \lambda_{i_n} & \\ & \lambda_{i_n}^{-1} \end{pmatrix}.$$

Denote $G_n(t) = P_n \Lambda(t) P_n^{-1}$ where

$$\Lambda(t)=\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}, \qquad t>0.$$

$G_n(t)$ is called the geodesics preserved by C_{i_n}. We have $C_{i_n}\cdot G_n(t)=G_n(t\lambda_{i_n})$.

There are $t_n>0$ and $A_n\in\Gamma$ s.t. $g_n=A_n\cdot G_n(t_n)\in F$. Since $\bar F$ is compact there is $\{g_{n_k}\}$ s.t. $g_{n_k}\to g\in\bar F\subset G$. We have

$$d(g_{n_k}, A_{n_k}C_{i_{n_k}}A_{n_k}^{-1}(g_{n_k}))=\log\lambda_{i_{n_k}}\to 0, \qquad k\to\infty.$$

This implies that g is not a point of discontinuity of Γ, which contradicts the fact that Γ is discrete. □

Denote

$$\begin{aligned}(48)\qquad &D(\beta,l)=\{x=(x_1,x_2)\in F\times F \mid \text{there are } z,\bar z\in O_{C(\varepsilon)}(e),\\ &0\leqq q,t\leqq l^\beta \text{ and } (e,e)\neq(A_1,A_2)\in\Gamma\times\Gamma \text{ s.t. (i)}\\ &A_1\cdot H_q(x_1\cdot z)=H_t(x_1)\cdot\bar z \text{ and (ii) } A_2\cdot H_q(x_2\cdot z)=\\ &H_t(x_2)\cdot\bar z\cdot\psi \text{ for some } \psi\in O(e;2Q/l)\}.\end{aligned}$$

It is clear that $pD(\beta,l)=E(\beta,l)\subset Y$ and $\tilde\nu(D(\beta,l))=\nu(E(\beta,l))$, $p\tilde\nu=\nu$, $\tilde\nu=\tilde\mu\times\tilde\mu$ where $\tilde\mu$ is the Haar measure in G, $\tilde\mu(F)=1$.

We have

$$H_q=\begin{pmatrix}1 & 0\\ q & 1\end{pmatrix} \qquad\text{and}\qquad H_q(g)=g\cdot H_q.$$

(i) and (ii) in (48) can be rewritten as follows:

$$x_1^{-1}\cdot A_1\cdot x_1=H_t\cdot\bar z\cdot H_q^{-1}\cdot z^{-1}=B(x_1,A_1), \tag{49}$$

$$x_2^{-1}\cdot A_2\cdot x_2=H_t\cdot\bar z\cdot\psi\cdot H_q^{-1}\cdot z^{-1}=B(x_1,A_1)\cdot\phi\in O(B(x_1,A_1),Q_1/l^{1-2\beta}) \tag{50}$$

$$\text{where } \phi\in O(e,Q_1/l^{1-2\beta})$$

for some $Q_1>0$ depending only on ε if l is sufficiently large.

Denote $\|B\|=\max|b_{ij}|$, $B=(b_{ij})$. Let $\Phi_1=\{A_1\in\Gamma/A_1$ satisfies (49) for some $x_1\in F$, $0\leqq q$, $t\leqq l^\beta$ and $z,\bar z\in O_{C(\varepsilon)}(e)\}$, $\Phi_2=\{A_2\in\Gamma/A_2$ satisfies (50) for some $x_2\in F$, $z,\bar z\in O_{C(\varepsilon)}(e)$, $0\leqq q$, $t\leqq l^\beta$ and $\psi\in O(e,2Q/l)\}$, $n_i=\operatorname{card}\Phi_i$, $i=1,2$.

Since $\bar F$ is compact, it follows from (49), (50) that if $A_1\in\Phi_1$ and $A_2\in\Phi_2$ then $\|A_1\|,\|A_2\|<Q_2l^{2\beta}$ for some $Q_2>0$ depending only on Γ and ε, if l is sufficiently large.

Since Γ is discrete we have

$$n_1,n_2<Q_3l^{6\beta} \tag{51}$$

where $Q_3>0$ depends only on Γ and ε.

Let us explore (49) and (50). One can see that since $z, \bar{z} \in O_{C(\varepsilon)}(e)$, $A_1, A_2 \neq -e$ if ε is sufficiently small.

If $A_1 = e$ in (i), then $A_2 = e$ satisfies (ii). Since Γ is discrete there is not any other $A_2 \in \Gamma$ satisfying (ii) if l is sufficiently large. In the same way, if $A_2 = e$ in (ii), then $A_1 = e$ in (i). So $(A_1, A_2) \neq (e, e)$ implies that $A_1 \neq e \neq A_2$.

Let $B \in G$, $A \in \Gamma$, $A \neq e, -e$ and let $f: G \to G$ $f(u) = uAu^{-1}$, $K(A, B) = K = \{u \in F / f(u) \in O(B, Q_1/l^{1-2\beta})\}$.

PROPOSITION 2. *There is $\tilde{R} > 0$ depending only on Γ and ε s.t. $\tilde{\mu}(K(A, B)) < \tilde{R}/l^{2(1-2\beta)}$ for all $B \in G$, $e, -e \neq A \in \Gamma$.*

PROOF. Let

$$A = P\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} P^{-1}$$

where λ is the eigenvalue of A with $|\lambda| > 1$. Denote

$$\Lambda(t) = P\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} P^{-1}, \qquad t > 0,$$

$\Lambda = \{\Lambda(t), t > 0\}$ is a one-parameter subgroup of G.

We have $f(\Lambda(t)) = A$ for all $t > 0$. Denote $X = G/\Lambda$ and let $X(K) = \{w \in X / w = u \cdot \Lambda$ for some $u \in K\}$.

Let

$$v_1 = v_1(e) = P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P^{-1} = \frac{d}{dt} P\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P^{-1}\Big|_{t=0}$$

and

$$v_2 = v_2(e) = P\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} P^{-1} = \frac{d}{dt} P\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} P^{-1}\Big|_{t=0}.$$

v_1, v_2 belong to the Lie algebra of G and $v_1(u) = u \cdot v_1$, $v_2(u) = u \cdot v_2$ belong to the tangent space T_u at $u \in G$.

Let us look at the differential

$$df(v_1(u)) = (uAu^{-1}) \cdot \frac{d}{dt}\left[(uA^{-1}u^{-1}) f\left(uP\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P^{-1}\right)\right]\Big|_{t=0} \in T_{uAu^{-1}}.$$

It is easy to compute that

$$\frac{d}{dt}\left[(uA^{-1}u^{-1}) f\left(uP\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P^{-1}\right)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left(uP\begin{pmatrix}\lambda^{-1} & \\ & \lambda\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}P^{-1}u^{-1}\right)\Bigg|_{t=0}$$

$$= uP\begin{pmatrix}0 & \lambda^{-2}-1\\ 0 & 0\end{pmatrix}P^{-1}u^{-1}$$

$$= (\lambda^{-2}-1)uv_1u^{-1}.$$

So we get $df(v_1(u)) = (uAu^{-1})\ (\lambda^{-2}-1)uv_1u^{-1}$.

In the same manner $df(v_2(u)) = (uAu^{-1})(\lambda^2-1)uv_2u^{-1}$.

Since $\bar{F}$ is compact there is $R_1 > 0$ depending only on Γ s.t. $\|uvu^{-1}\| \geqq R_1\|v\|$ for all v from the Lie algebra of G and all $u \in \bar{F}$.

Using Proposition 1 we have

$$\|df(v_1(u))\| \geqq R_2\|v_1(u)\|,$$
(52)
$$\|df(v_2(u))\| \geqq R_2\|v_2(u)\|,$$

where

$$R_2 = R_1 \cdot \left(\frac{\tau^2-1}{\tau^2}\right) > 0, \qquad \tau > 1.$$

$(v_1(u), v_2(u))$ can be thought of as a basis in the tangent space T_w of X, $w = u \cdot \Lambda$.

Let $\bar{f}$: $X \to G$ be $\bar{f}(w) = f(u)$ if $w = u \cdot \Lambda$. $\bar{f}$ is a diffeomorphism from X onto the range $\bar{f}$ which is a two-dimensional surface in G. Let dm denote the measure on the range $\bar{f}$ generated by the Riemannian metric on range $\bar{f}$ and let $d\bar{\mu}$ be the measure on $X = G/\Lambda$ generated by $\bar{\mu}$.

Let $\bar{f}_K$: $X(K) \to G$ be the restriction of $\bar{f}$ on $X(K)$ and let

$$\bar{R} = \text{range}\,\bar{f}_K \subset O(B, Q_1/l^{1-2\beta}) \cap f(G)$$
$$= \{B\cdot\phi \mid \phi \in O(e, Q_1/l^{1-2\beta}), \operatorname{Tr} B\cdot\phi = \operatorname{Tr} A\} = S.$$

One can see that $m(S) \leqq \bar{Q}/l^{2(1-2\beta)}$ for some $\bar{Q} > 0$.

We have

$$\bar{Q}/l^{2(1-2\beta)} \geqq m(\bar{R}) = \int_{X(K)} |J(w)|\,d\bar{\mu}(w)$$

where $J(w)$ is the Jacobian of $\bar{f}$ at $w \in X(K)$.

(52) says that

$$|J(w)| \geqq R_3\ w \in X(K) \qquad \text{where } R_3 > 0 \text{ depends only on } \Gamma.$$

We get

$$\bar{Q}/l^{2(1-2\beta)} \geqq R_3\bar{\mu}(X(K))$$

or

$$\bar{\mu}(X(K)) \leqq R_4/l^{2(1-2\beta)}, \qquad R_4 > 0.$$

Since $\bar{F}$ is compact this implies that $\bar{\mu}(K) \leqq \tilde{R}/l^{2(1-2\beta)}$ for some $\tilde{R} > 0$ depending only on Γ and ε. □

Proof of Lemma 5. Let $D(\beta, l)$ be as in (48). We have to prove that there are $\beta_0, N, \theta, R > 0$ s.t. if $0 < \beta \leqq \beta_0$, $l \geqq N$ then $\bar{\nu}(D(\beta, l)) < Rl^{-\theta}$.

Denote $D(x_1, \beta, l) = \{x_2 \in G/(x_1, x_2) \in D(\beta, l)\}$. We have $D(x_1, \beta, l) \subset \bigcup K(A_2, x_1^{-1}A_1x_1)$ where the union is taken over all $e, -e \neq A_1 \in \Phi_1$, $e, -e \neq A_2 \in \Phi_2$ (see (51)).

Using Proposition 2 and (51) we have $\bar{\mu}(D(x_1, \beta, l)) \leqq Rl^{-(2-10\beta)}$ for some $R > 0$, if l is sufficiently large, say $l \geqq N$ for some $N > 0$.

This implies that $\bar{\nu}(D(\beta, l)) \leqq Rl^{-(2-10\beta)}$ if $l \geqq N$. We complete the proof setting $\beta_0 = 1/10$, $\theta = 1$. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIF. 94720 USA